Analytic Pricing of Interest Rate Swap Insurance

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Abstract

We seek analytic formulae for the pricing of non-payment insurance on a vanilla interest rate swap. In the event of a default by the swap counterparty, the net PV of all future cash flows, floored at a given strike level (typically zero), is required to be paid by the insurer to the insured party at the time of default. We suppose that the domestic rates and the counterparty default intensity are stochastic and potentially correlated. The credit intensity model is taken to be Black-Karasinski and the interest rate model to be beta-blend, a family of models which encompasses within its scope both the Black-Karasinski and the Hull-White cases. A Green's function solution for the governing PDE is found as a perturbation expansion valid in the limit of low interest rates and credit default intensities. This is used to calculate to first order accuracy the fair price of insuring payer or receiver swaps, including the impact of capping the insurance protection amount.

1 Introduction

We seek analytic formulae for the pricing of non-payment insurance on a vanilla interest rate swap. We suppose in the first instance that the insurance buyer receives floating rate payments from the swap counterparty and pays fixed. In the event of a default by the swap counterparty (floating rate payer)

before some specified time T, the net PV of all future cash flows, floored at a given strike level (typically zero), is required to be paid by the insurer to the insured party at the time of default. The approach taken in the calculation is similar to that of Turfus (2017a) where the cost of insuring an equity option was considered. Use is made below of the methodology set out in the work of Horvath et al. (2017) and Turfus (2017b).

We start in section 2 by defining the underlying processes in terms of suitably chosen auxiliary variables and the no-arbitrage conditions they are required to satisfy; we infer therefrom the governing PDE for contingent claims. Given the intractability of this PDE, we define in section 3 an asymptotic scaling of the underlying variables and associated functions based on the assumed smallness of the two short rates: interest rate and credit default intensity. This gives rise to a tractable leading order PDE with the "intractable" parts isolated as a perturbation: we can then seek solutions as perturbation series. Rather than looking to obtain particular solutions directly, a Green's function for the full PDE is sought as a perturbation expansion. The constituent cash flows associated with the interest rate swap are valued conditional on a default at time $\tau < T$ in section 4 in terms of the asymptotically scaled variables. The sum of these contributions, if positive, is the exposure payment which must be made by the insurer. The further calculation required to derive the fair value of insurance for each of the constituent conditional cash flows is presented in Appendix A. The contributions are summed and the aggregate price of insuring all cash flows associated with payer and receiver swaps is presented in section 5, including specialised results valid in the (singular) Hull-White limit. Consideration is also given to assessing the impact of a cap on the insurance payment amount. Finally the predictions of the asymptotic model are compared with results obtained from a Monte Carlo simulation in section 6.

2 Stochastic Modelling

2.1 Definition of Underlying Processes

Our modelling approach is to represent the interest rate and the credit default intensity as short rate diffusion processes. Specifically, we suppose the interest rate process r_t to be governed for $t > 0$ by a beta-blend short rate model as set out in Turfus (2016), Horvath et al. (2017) and Turfus (2017b). This encompasses both the Hull & White (1990) and Black & Karasinski (1991) short rate models. We suppose the credit default intensity process λ_t to be governed by a Black-Karasinski (lognormal) short rate model.

We shall find it convenient to work with auxiliary variables $\hat{x}_t$ and $\hat{y}_t$ satisfying the following Ornstein-Uhlenbeck processes:

$$d\hat{x}_t = -\alpha_r \hat{x}_t dt + \sigma_r(t)\, d\tilde{W}_t^1, \tag{2.1}$$

$$d\hat{y}_t = -\alpha_\lambda \hat{y}_t dt + \sigma_\lambda(t)\, dW_t^2, \tag{2.2}$$

where $d\tilde{W}_t^1$ and dW_t^2 are Brownian motions under the usual equivalent martingale measure with

$$\mathrm{corr}(\tilde{W}_t^1, W_t^2) = \rho_{r\lambda}$$

assumed and $\hat{x}_0 = \hat{y}_0 = 0$. These auxiliary variables are related to the interest rate r_t and the credit default intensity λ_t, respectively, by

$$(1-\beta)\, r_t + \beta\bar{r}(t) = (\bar{r}(t) + (1-\beta)r^*(t))\mathcal{E}\left(\frac{(1-\beta)\hat{x}_t}{|\bar{r}(t)|^\beta}\right), \tag{2.3}$$

$$\lambda_t = (\bar{\lambda}(t) + \lambda^*(t))\mathcal{E}(\hat{y}_t), \tag{2.4}$$

with $\beta \in [0,1)$ assumed. Here $\bar{r}(t)$ is the instantaneous forward interest rate and $\bar{\lambda}(t)$ the associated credit spread (see (2.9) below), with $\sigma_r(t)$ and $\sigma_\lambda(t)$ their respective volatilities and α_r and α_λ their mean reversion rates. Here $\mathcal{E}(X_t) := \exp\left(X_t - \frac{1}{2}[X]_t\right)$ is a stochastic exponential with $[X]_t$ the quadratic variation of a process X_t under the requisite measure. We note here specifically that $[\hat{x}]_t = I_r(t)$ where we define

$$I_r(t) := \int_0^t e^{-2\alpha_r(t-u)}\sigma_r^2(u)\, du. \tag{2.5}$$

The required form of the configurable functions $r^*(t)$ and $\lambda^*(t)$ is determined by calibration of the model to satisfy the no-arbitrage conditions set out below. We further assume that $\hat{x}_0 = \hat{y}_0 = 0$, with $t = 0$ the "as of" date for which the model is calibrated. As can be seen, the beta-blend model for the interest rate represents a hybrid between the Hull-White model ($\beta \uparrow 1$) and the Black-Karasinski model ($\beta = 0$).

The no-arbitrage conditions

The formal no-arbitrage constraints which determine the functions $r^*(t)$ and $\lambda^*(t)$ are as follows. First, by considering a risk-free cash flow at time t, we deduce

$$E\left[e^{-\int_0^t r_s\,ds}\right] = D(0,t), \tag{2.6}$$

for $0 < t \leq T_m$, where T_m is the longest maturity date for which the model is calibrated, and

$$D(t_1,t_2) = e^{-\int_{t_1}^{t_2}\overline{r}(s)\,ds} \tag{2.7}$$

is the t_1-forward price of the t_2-maturity zero coupon bond. Likewise, considering a *risky* cash flow, we have

$$E\left[e^{-\int_0^t (r_s+\lambda_s)\,ds}\right] = B(0,t), \tag{2.8}$$

where

$$B(t_1,t_2) = e^{-\int_{t_1}^{t_2}(\overline{r}(s)+\overline{\lambda}(s))\,ds} \tag{2.9}$$

is the t_1-forward price of a risky (defaultable) cash flow. We shall assume the bond prices can be ascertained at the initial time $t = 0$ from the market, whence we can view (2.7) and (2.9) as implicitly *defining* $\overline{r}(t)$ and $\overline{\lambda}(t)$.

2.2 Derivation of Governing PDE

We consider the problem of pricing a security with maturity T which, in the event of a counterparty default at some stopping time $\tau \in (0,T]$, pays a (domestic currency) cash amount which depends on $\hat{\xi} := \hat{x}_\tau$ (but notably not on $\hat{y}_\tau$), denoting this payoff by $\hat{P}(\hat{\xi},\tau)$. We further introduce the convenient shorthand notation that, for a process X_t and deterministic function $f(.)$,

$$\mathcal{E}_x(f(t)X_t)) := \mathcal{E}(f(t)X_t))|_{X_t=x},$$

in terms of which we can re-write (2.3) and (2.4) as $r_t = r(\hat{x}_t,t)$ and $\lambda_t = \lambda(\hat{y}_t,t)$, where

$$r(\hat{x},t) := \frac{1}{1-\beta}\left((\overline{r}(t) + (1-\beta)r^*(t))\mathcal{E}_{\hat{x}}\left(\frac{(1-\beta)\hat{x}_t}{|\overline{r}(t)|^\beta}\right) - \beta\overline{r}(t)\right), \tag{2.10}$$

$$\lambda(\hat{y},t) := (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}_{\hat{y}}(\hat{y}_t), \tag{2.11}$$

Writing the price of the security at time $t \in [0, \tau \wedge T]$ as $f_t = \hat{f}(\hat{x}_t,\hat{y}_t,t)$, we can infer by application of the Feynman-Kac theorem to (2.1) and (2.2) in the standard manner that the function $\hat{f}(\hat{x},\hat{y},t)$ satisfies the following backward diffusion equation:

$$\left(\frac{\partial}{\partial t} + \hat{\mathcal{L}} - \overline{r}(t) - \overline{\lambda}(t)\right)\hat{f}(\hat{x},\hat{y},t) = -\lambda(\hat{y},t)\hat{P}(\hat{x},t), \tag{2.12}$$

subject to $\hat{f}(\hat{x},\hat{y},T) = 0$, where

$$\begin{aligned}\hat{\mathcal{L}} := -\alpha_r\hat{x}\frac{\partial}{\partial\hat{x}} - \alpha_\lambda\hat{y}\frac{\partial}{\partial\hat{y}} + \frac{1}{2}\left(\sigma_r^2(t)\frac{\partial^2}{\partial\hat{x}^2} + 2\rho_{r\lambda}\sigma_r(t)\sigma_\lambda(t)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}} + \sigma_\lambda^2(t)\frac{\partial^2}{\partial\hat{y}^2}\right) - (r(\hat{x},t) - \overline{r}(t))\cdot \\ - (\lambda(\hat{y},t) - \overline{\lambda}(t))\cdot\end{aligned} \tag{2.13}$$

In the absence of closed form solutions to (2.12) and guided by the work of Hagan et al. (2015), Horvath et al. (2017) and Turfus (2017b), we propose a perturbation expansion approach as follows.

3 Asymptotic Modelling

For both short rate models we formally apply a "low rates" assumption. To this end we define small parameters

$$\epsilon_r := \frac{1}{\alpha_r T_m} \int_0^{T_m} \overline{r}(t)\, dt \tag{3.1}$$

$$\epsilon_\lambda := \frac{1}{\alpha_\lambda T_m} \int_0^{T_m} \overline{\lambda}(t)\, dt \tag{3.2}$$

and $O(1)$ functions

$$\begin{aligned} \tilde{r}(t) &:= \epsilon_r^{-1} \frac{\overline{r}(t)}{1-\beta}, \\ \tilde{r}^*(t) &;= \epsilon_r^{-1} r^*(t) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \tilde{\lambda}(t) &= \epsilon_\lambda^{-1} \overline{\lambda}(t), \\ \tilde{\lambda}^*(t) &= \epsilon_\lambda^{-1} \lambda^*(t). \end{aligned} \tag{3.4}$$

Note, we make no formal requirement that either $\sigma_r(t)$ or $\sigma_\lambda(t)$ be small (scaled, say, in terms of their respective mean reversion rates). But, if they are, this will in general improve the accuracy of the perturbation expansions deduced. We further define a new scaled variable x_t and associated scaled volatility $\sigma_x(t)$ by

$$\begin{aligned} x_t &:= \epsilon_r^{-\beta} \hat{x}_t e^{\alpha_r t}, \\ \sigma_x(t) &= \epsilon_r^{-\beta} \sigma_r(t) e^{\alpha_r t}. \end{aligned} \tag{3.5}$$

Likewise we define

$$\begin{aligned} y_t &:= \hat{y}_t e^{\alpha_\lambda t}, \\ \sigma_y(t) &:= \sigma_\lambda(t) e^{\alpha_\lambda t}. \end{aligned} \tag{3.6}$$

Here the exponential time scaling is to facilitate removal of the mean reverting drift terms in (2.13). We further define new functional forms $f(\cdot)$ and $P(\cdot)$ by:

$$\begin{aligned} f(x_t, y_t, t) &:= \epsilon_\lambda^{-1} \hat{f}(\hat{x}_t, \hat{y}_t, t), \\ P(x_\tau, \tau) &:= \hat{P}(\hat{x}_\tau, \tau), \end{aligned}$$

where $\hat{x}_t$ and $\hat{y}_t$ are related to x_t and y_t by (3.5) and (3.6), respectively for $t \in [0, \tau \wedge T]$. In this notation, (2.1) and (2.2) can be re-expressed as

$$dx_t = \sigma_x(t)\, dW_t^1, \tag{3.7}$$

$$dy_t = \sigma_y(t)\, dW_t^2 \tag{3.8}$$

and (2.12) as

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - \overline{r}(t) - \overline{\lambda}(t) - \epsilon_r h(x,t) - \epsilon_\lambda g(y,t) \right) f(x,y,t) = -(\tilde{\lambda}(t) + g(y,t)) P(x,t) \tag{3.9}$$

where

$$\mathcal{L}[\cdot\,] := \frac{1}{2}\left(\sigma_x^2(t) \frac{\partial^2}{\partial x^2} + 2\rho_{r\lambda}\sigma_x(t)\sigma_y(t) \frac{\partial^2}{\partial x \partial y} + \sigma_y^2(t) \frac{\partial^2}{\partial y^2} \right) \tag{3.10}$$

is a pure diffusion and

$$h(x,t) := h(x,t,t), \tag{3.11}$$
$$g(y,t) := g(y,t,t), \tag{3.12}$$

with

$$h(x,t,t_1) := (\tilde{r}(t_1) + \tilde{r}^*(t_1))\mathcal{E}_x\left(F_\beta(t_1)x_t\right) - \tilde{r}(t_1), \quad t_1 \geq t, \tag{3.13}$$
$$F_\beta(t) := \frac{(1-\beta)^{1-\beta}e^{-\alpha_r t}}{|\tilde{r}(t)|^\beta}, \tag{3.14}$$
$$g(y,t,t_1) := (\tilde{\lambda}(t_1) + \tilde{\lambda}^*(t_1))\mathcal{E}_y\left(e^{-\alpha_\lambda t_1}y_t\right) - \tilde{\lambda}(t_1), \quad t_1 \geq t. \tag{3.15}$$

We seek a Green's function solution for (3.9) as a joint power series in ϵ_r and ϵ_λ. The reader is referred to Turfus (2017b) for details of the derivation up to second order accuracy. We note here the important observation there made that satisfaction of the no-arbitrage conditions requires that $\tilde{r}^*(t) \sim \epsilon_r r_1^*(t)$ and $\tilde{\lambda}^*(t) \sim \epsilon_r \lambda_{1,0}^*(t) + \epsilon_\lambda \lambda_{0,1}^*(t)$ with errors $\mathcal{O}(\epsilon_r^2)$ and $\mathcal{O}(\epsilon_r^2 + \epsilon_\lambda^2)$, respectively, where

$$r_1^*(t) = \tilde{r}(t)\int_0^t \tilde{r}(u)\left(\exp\left(F_\beta(u)F_\beta(t)I_x(0,u)\right) - 1\right)du, \tag{3.16}$$
$$\lambda_{0,1}^*(t) = \tilde{\lambda}(t)\int_0^t \tilde{\lambda}(u)\left(\exp\left(e^{-\alpha_\lambda(t+u)}I_y(0,u)\right) - 1\right)du. \tag{3.17}$$
$$\begin{aligned}\lambda_{1,0}^*(t) = \tilde{r}(t)\int_0^t \tilde{\lambda}(u)\left(\exp\left(e^{-\alpha_\lambda u}F_\beta(t)I_{xy}(0,u)\right) - 1\right)du \\ + \tilde{\lambda}(t)\int_0^t \tilde{r}(u)\left(\exp\left(e^{-\alpha_\lambda t}F_\beta(u)I_{xy}(0,u)\right) - 1\right)du,\end{aligned} \tag{3.18}$$

with

$$I_x(t_1,t_2) := \int_{t_1}^{t_2} \sigma_x^2(u)\,du \tag{3.19}$$
$$I_y(t_1,t_2) := \int_{t_1}^{t_2} \sigma_y^2(u)\,du. \tag{3.20}$$
$$I_{xy}(t_1,t_2) := \rho_{r\lambda}\int_{t_1}^{t_2} \sigma_x(u)\sigma_y(u)\,du. \tag{3.21}$$

Other details we shall report as needed.

4 Exposure Modelling

As stated in §2.2 above, the insurance payoff is determined by the expected value of all future cash flows associated with the derivative insured at the time of a counterparty default, floored at zero; in other words the positive exposure. These flows are of two types:

1. fixed coupon flows, and
2. Libor flows.

To obtain an expression representing $P(x,t)$ in (3.9), we need to evaluate these cash flows at arbitrary times $t \in (0,T)$, independently of whether the counterparty defaults or not.

4.1 Conditional Pricing of Component Flows

Fixed Coupon Flow

We consider the PV as of time $t < T$ of a fixed flow of one unit of domestic currency at time T, denoting this as $V_F^T(x_t, t)$. Turfus (2017b) finds that

$$V_F^T(x,t) \sim D(t,T)\left(1 + \epsilon_r F_{1,0}^T(x,t) + \epsilon_r^2 F_{2,0}^T(x,t)\right) \tag{4.1}$$

with $O(\epsilon_r^3)$ errors, where

$$\begin{aligned}
F_{1,0}^T(x,t) &:= -\int_t^T \tilde{r}(u)(\mathcal{E}_x(F_\beta(u)x_t) - 1)\, du \\
F_{2,0}^T(x,t) &:= \tfrac{1}{2}F_{1,0}^2(x,t) + \int_t^T \mathcal{E}_x(F_\beta(v)x_t) \\
&\qquad \left(\tilde{r}(v)\int_t^v \tilde{r}(u)\mathcal{E}_x(F_\beta(u)x_t)\left(\exp\left(F_\beta(u)F_\beta(v)I_x(0,u)\right) - 1\right) du - r_1^*(v)\right) dv.
\end{aligned}$$

We shall make use here only of the first order expression, writing

$$V_F^{t_i}(x,t) = D(t,t_i)\left(1 + \epsilon_r F_{1,0}^{t_i}(x,t)\right) + \mathcal{O}(\epsilon_r^2), \quad t < t_i. \tag{4.2}$$

Libor Flow

Let us next denote the PV as of time $t < t_i$ of the Libor flow fixing at t_{i-1} and paying at time t_i by $V_L^{(i)}(x_t,t)$ per unit notional. This can be expressed as

$$V_L^{(i)}(x,t) = \begin{cases} V_F^{t_{i-1}}(x,t) - V_F^{t_i}(x,t), & t < t_{i-1} \\ V_F^{t_i}(x,t)\left(V_F^{t_i}(x,t_{i-1})^{-1} - 1\right), & t_{i-1} \le t < t_i, \end{cases} \tag{4.3}$$

the latter expression being required to calculate the impact of a future Libor payment which has already fixed at the time of counterparty default. Noting that $D(t_{i-1},t_i)^{-1} - 1$ is in fact first order in ϵ_r, the value of the Libor payment can be rewritten as

$$V_L^{(i)}(x,t) = \begin{cases} D(t,t_{i-1})\left(1 + \epsilon_r F_{1,0}^{t_{i-1}}(x,t)\right) - D(t,t_i)\left(1 + \epsilon_r F_{1,0}^{t_i}(x,t)\right) + \mathcal{O}(\epsilon_r^2), & t < t_{i-1} \\ D(t_{i-1},t)^{-1}\frac{1+\epsilon_r F_{1,0}^{t_i}(x,t)}{1+\epsilon_r F_{1,0}^{t_i}(x,t_{i-1})} - D(t,t_i)\left(1 + \epsilon_r F_{1,0}^{t_i}(x,t)\right) + \mathcal{O}(\epsilon_r^2), & t_{i-1} \le t < t_i. \end{cases} \tag{4.4}$$

4.2 Aggregating the Payoff Flows

Next we compute the positive exposure function $P(x,t)$ appearing in (3.9) above using the expressions for the component flows derived in the previous subsection. We consider for simplicity that the coupon flows occur at the same dates and the day count convention used is the same on both legs.[1] Suppose that the payment periods are $[t_{i-1}, t_i]$, for $i = 1, \ldots, n$ and the day count fractions Δ_i. Consider a payer swap where the pay leg has a fixed coupon amount c and the other leg receives Libor.

In terms of the above notation the value of the positive exposure at default time $\tau < T$ is then given by

$$P(x,\tau) = \max\left\{\sum_{i=1}^n \left(V_L^{(i)}(x,\tau) - c\Delta_i V_F^{t_i}(x,\tau)\right)\mathbb{1}_{\tau<\min\{t_i,T\}}, 0\right\} \tag{4.5}$$

[1] It is not difficult to modify the formulae derived to take account of a difference in day count convention between the two legs.

per unit notional. It remains to solve (3.9) for $f(x, y, t)$ subject to (4.5) and a homogeneous final condition

$$f(x, y, T) = 0, \tag{4.6}$$

to the desired order of accuracy in the parameters ϵ_r and ϵ_λ. In practice we will restrict our attention to first order in each, to which end we will need only the leading order representation of the Green's function derived in Turfus (2017b), in combination with first order versions of the expressions derived in section 4.1 above. This calculation is performed in Appendix A below, the main conclusions of which are set out in the following section.

4.3 Aggregated Payoff for Hull-White Case

For future use, we express the exposure expression in (4.5) in original unscaled variables for the particular case of a Hull-White interest rate model. We obtain straightforwardly, in self-evident notation,

$$\text{Exposure} = \sum_{i=1}^{n} \left(\hat{V}_L^{(i)}(\hat{x}, \tau) - c\Delta_i \hat{V}_F^{t_i}(\hat{x}, \tau) \right), \tag{4.7}$$

where, using the first order approximation for the fixed flow values as of time $t < t_i$, we have

$$\hat{V}_F^{t_i}(\hat{x}, t) \sim D(t, t_i) \left(1 - \hat{x}B^*(t_i - t)\right), \tag{4.8}$$

with errors $= \mathcal{O}(\epsilon_r^2)$, where

$$B^*(v) := \frac{1 - e^{-\alpha_r v}}{\alpha_r}. \tag{4.9}$$

Likewise we have to the same level of accuracy

$$\hat{V}_L^{(i)}(\hat{x}, t) \sim \begin{cases} D(t, t_{i-1}) \left(1 - \hat{x}B^*(t_{i-1} - t)\right) - D(t, t_i) \left(1 - \hat{x}B^*(t_i - t)\right), & t \leq t_{i-1} \\ D(t_{i-1}, t)^{-1} \frac{1 - \hat{x}B^*(t_i - t)}{1 - \hat{x}B^*(t_i - t_{i-1})} - D(t, t_i) \left(1 - \hat{x}B^*(t_i - t)\right), & t_{i-1} < t < t_i. \end{cases} \tag{4.10}$$

5 Main Results

PV Insurance for Payer Swap

We conclude from the asymptotic analysis that, with relative error $= \mathcal{O}(\epsilon_r + \epsilon_\lambda)$, the cost of insurance purchased at $t = 0$ on a payer swap is given in original notation by

$$V_{\text{insurance}} \sim \sum_{i=1}^{n} \left(\hat{f}_L^{(i)} - c\hat{f}_F^{t_i} \Delta_i \right), \tag{5.1}$$

where

$$\hat{f}_F^w := \int_0^{w \wedge T} \overline{\lambda}(v) B(0, v) D(v, w) N(-\hat{d}_1(\xi^*(v), v)) \, dv, \tag{5.2}$$

$$\begin{aligned} \hat{f}_L^{(i)} := \left(D(t_{i-1}, t_i)^{-1} - 1\right) \hat{f}_F^{t_i} + D(t_{i-1}, t_i)^{-1} \int_0^{t_i \wedge T} \overline{\lambda}(v) B(0, v) D(v, t_i) \int_{t_{i-1}}^{t_i} \frac{\overline{r}(u)}{1 - \beta} \\ \left(\exp\left(e^{-\alpha_\lambda(v - v \wedge t_{i-1})} \hat{F}_\beta(u, v \wedge t_{i-1}) I_{r\lambda}(v \wedge t_{i-1})\right) N\left(-\hat{d}_2(\xi^*(v), u, v, v \wedge t_{i-1})\right) \right. \\ \left. - N\left(-\hat{d}_1(\xi^*(v), v)\right) \right) du \, dv, \end{aligned} \tag{5.3}$$

with

$$I_{r\lambda}(t) := \rho_{r\lambda} \int_0^t e^{-(\alpha_r+\alpha_\lambda)(t-u)} \sigma_r(u)\sigma_\lambda(u)\, du, \tag{5.4}$$

$$\hat{d}_1(x,v) := \frac{x - I_{r\lambda}(v)}{\sqrt{I_r(v)}}, \tag{5.5}$$

$$\hat{d}_2(x,u,v,w) := \hat{d}_1(x,v) - \frac{\hat{F}_\beta(u, 2w-v) I_r(w)}{\sqrt{I_r(v)}}, \tag{5.6}$$

$$\hat{F}_\beta(u,v) := \frac{(1-\beta)e^{-\alpha_r(u-v)}}{|\overline{r}(u)|^\beta}, \quad u \geq v, \tag{5.7}$$

$$\xi^*(v) := e^{-\alpha_r v} x^*(v), \tag{5.8}$$

where $x^*(v)$ is given by (A.2). Alternatively, in the Hull-white case, $\xi^*(v)$ is given to leading order as the zero of (4.7) with $\tau = v$, subject to (4.8) and (4.10).

Note that in the Hull-White case ($\beta = 1$), we must take the limit as $\beta \to 1$ in the inner integral in (5.3). Applying l'Hôpital's theorem and computing explicitly the inner integration w.r.t. u, we obtain:

$$\hat{f}_L^{(i)} \to \left(D(t_{i-1},t_i)^{-1} - 1\right) \hat{f}_F^{t_i} + D(t_{i-1},t_i)^{-1} \int_0^{t_i \wedge T} \overline{\lambda}(v) B(0,v) D(v,t_i) B^*(t_i - t_{i-1})$$
$$\left(\gamma(t_{i-1},v) I_{r\lambda}(v \wedge t_{i-1}) N(-\hat{d}_1(\xi^*(v),v)) + e^{-\alpha_r|v-t_{i-1}|} I_r(v \wedge t_{i-1}) \frac{N'(-\hat{d}_1(\xi^*(v),v))}{\sqrt{I_r(v)}}\right) dv, \tag{5.9}$$

with

$$\gamma(u,v) := \begin{cases} e^{-\alpha_\lambda(v-u)}, & u \leq v, \\ e^{-\alpha_r(u-v)}, & u > v. \end{cases} \tag{5.10}$$

This completes our leading order calculation. We note in parting that the sum of $\hat{f}_F^{t_i}$ in (5.1) can be computed more efficiently in practice by refactoring as

$$\sum_{i=1}^n \hat{f}_F^{t_i} = \sum_{i=1}^n \sum_{j=1}^i D(t_j,t_i) \int_{t_{j-1}\wedge T}^{t_j \wedge T} \overline{\lambda}(v) B(0,v) D(v,t_j) N(-\hat{d}_1(\xi^*(v),v))\, dv, \tag{5.11}$$

where we define $t_0 \equiv 0$. The first term in (5.9) can be handled identically. As the integrand is now independent of i, it effectively only needs to be integrated once, from 0 to T.

PV Insurance for Receiver Swap

The corresponding expression for the receiver swap is easily derived. (5.1) is replaced straightforwardly by

$$V_{\text{insurance}} \sim \sum_{i=1}^n \left(c\hat{f}_F^{t_i} \Delta_i - \hat{f}_L^{(i)}\right) \tag{5.12}$$

with the caveat that $-\hat{d}_i(\cdot)$ is replaced by $\hat{d}_i(\cdot)$ in (5.2) and (5.3). In the Hull-White case we have

$$\hat{f}_L^{(i)} \sim \left(D(t_{i-1},t_i)^{-1} - 1\right) \hat{f}_F^{t_i} + D(t_{i-1},t_i)^{-1} \int_0^{\min\{t_i,T\}} \overline{\lambda}(v) B(0,v) D(v,t_i) B^*(t_i - t_{i-1})$$
$$\left(\gamma(t_{i-1},v) I_{r\lambda}(v \wedge t_{i-1}) N(\hat{d}_1(\xi^*(v),v)) - e^{-\alpha_r|v-t_{i-1}|} I_r(v \wedge t_{i-1}) \frac{N'(\hat{d}_1(\xi^*(v),v))}{\sqrt{I_r(v)}}\right) dv. \tag{5.13}$$

PV Insurance with Capped Payment

We consider finally how to handle the situation where the insurance payment is capped at a given level, K say. We deal with the case of a payer swap but the calculation is exactly analogous for a receiver swap. Also, although we assume that K is constant for notational convenience, it is a straightforward matter to modify (5.14) and (5.15) below to take account of a time-dependent cap. First let us consider the modified form of the payoff if the critical boundary were at K rather than 0, viz.

$$P_K(x,\tau) = \max\left\{ \sum_{i=1}^{n} \left(V_L^{(i)}(x,\tau) - c\Delta_i V_F^{t_i}(x,\tau) \right) \mathbb{1}_{\tau<t_i} - K, 0 \right\} \tag{5.14}$$

Further, let us denote the critical value of x at which this payoff comes into the money by $x_K^*(\tau)$ and the corresponding value of $\hat{x}$ by $\xi_K^*(\tau)$. Let the corresponding PV of fixed and floating payments paid conditional on this critical value being exceeded upon counterparty default be written as $\hat{f}_{F,K}^{t_i}$ and $\hat{f}_{L,K}^{(i)}$, respectively. It remains to value the additional effective payment of K at default in this version of the contract. The calculation is almost identical to that for a fixed cash flow at T except in that the cash flow is instead brought forward to τ. We obtain the following leading order estimate:[2]

$$\hat{f}_K^T = K \int_0^T \overline{\lambda}(v) B(0,v) N(-\hat{d}_1(\xi_K^*(v), v))\, dv \tag{5.15}$$

and conclude

$$V_{\text{insurance}}^K \sim \sum_{i=1}^{n} \left(\hat{f}_{L,K}^{(i)} - c\hat{f}_{F,K}^{t_i} \Delta_i \right) - \hat{f}_{F,K}^T, \tag{5.16}$$

Finally we observe that the required payoff for an insurance payment capped at K is $P(x,\tau) - P_K(x,\tau)$, whence, by the principle of portfolio replication, the PV of the capped insurance will be

$$V_{\text{ins\_capped}} = V_{\text{insurance}} - V_{\text{insurance}}^K. \tag{5.17}$$

6 Comparison with Monte Carlo Results

Comparisons were made between prices based on our formulae and those generated by a Monte Carlo representation of the same model for the following base situation. Parameters were varied in the manner specified to see their impact on pricing and on the accuracy of our approximation. All prices below are reported in M EUR.

Product details	
Swap maturity	10y
Swap coupons	Receiving quarterly 2.5% fixed; paying 6M LIBOR + 40bp spread
Notional	100M EUR
Protection period	6y
Insurance coupon rate	Paying semi-annually 4%
Market details	
Interest rate	Swap rates increase from -24bp to 81bp during the lifetime of the swap
Short rate rol (normal)	$\sigma_r(t)$ increases from 0.17% to 0.81% during the insured period with $\alpha_r = 25\%$
Market CDS rate	Flat 3.5% with recovery rate 55%
Credit vol (lognormal)	$\sigma_\lambda(t) = 70\%$ with $\alpha_\lambda = 30\%$
Correlation	$\rho_{r\lambda} = 30\%$.

[2] If we consider K to be $\mathcal{O}(\epsilon_r)$, the level of accuracy of this expression will be the same as those previously computed. But even if in practice K is chosen to be relatively large, this will usually not be too much of a problem since the payoff associated with the cap will then typically be significantly out of the money and so constitute only a small correction to the uncapped value.

The impact of interest rate volatility in the absence of correlation is illustrated in Fig. 1. Here we expect no impact of credit volatility. A relative shift of the entire volatility curve is used: -100% corresponds to no volatility, while +100% represents a doubling of the volatility levels. As can be seen, the asymptotic model predictions follow the Monte Carlo results closely. However it is notable that there remains a residual difference between the models even when the volatility is zero, at which point the asymptotic model becomes exact. This difference is a consequence of an approximation used in the Monte Carlo model, which does not model default events continuously but at discrete weekly intervals. This approximation results in a small underestimate of the value of protection. We conclude that our asymptotic formula is calculating the PV of the insurance correct to within a basis point of notional.

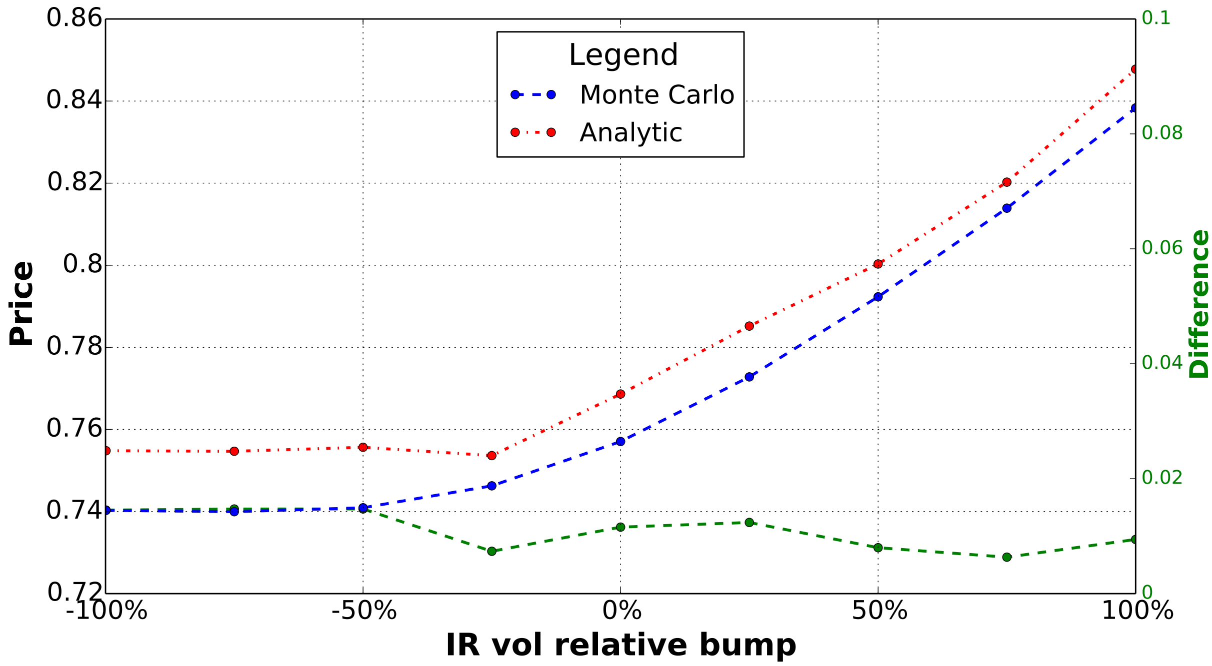

Figure 1: PV dependence of interest rate swap insurance on interest rate volatility level.

The impact of correlation on prices is illustrated in Fig. 2. The small discrepancy of around 1 bp of notional at zero correlation is again evident. The agreement for non-zero correlation is generally good, at all times within 6 bp of notional even in the most unfavourable case with a correlation of -100%. But there is a clear trend for the asymptotic formula to predict a greater impact of correlation than is suggested by the Monte Carlo simulations. This difference may be due to higher order terms which have not been calculated, specifically at $\mathcal{O}(\epsilon_r \epsilon_\lambda^2)$. More detailed calculation is required to resolve the issue.

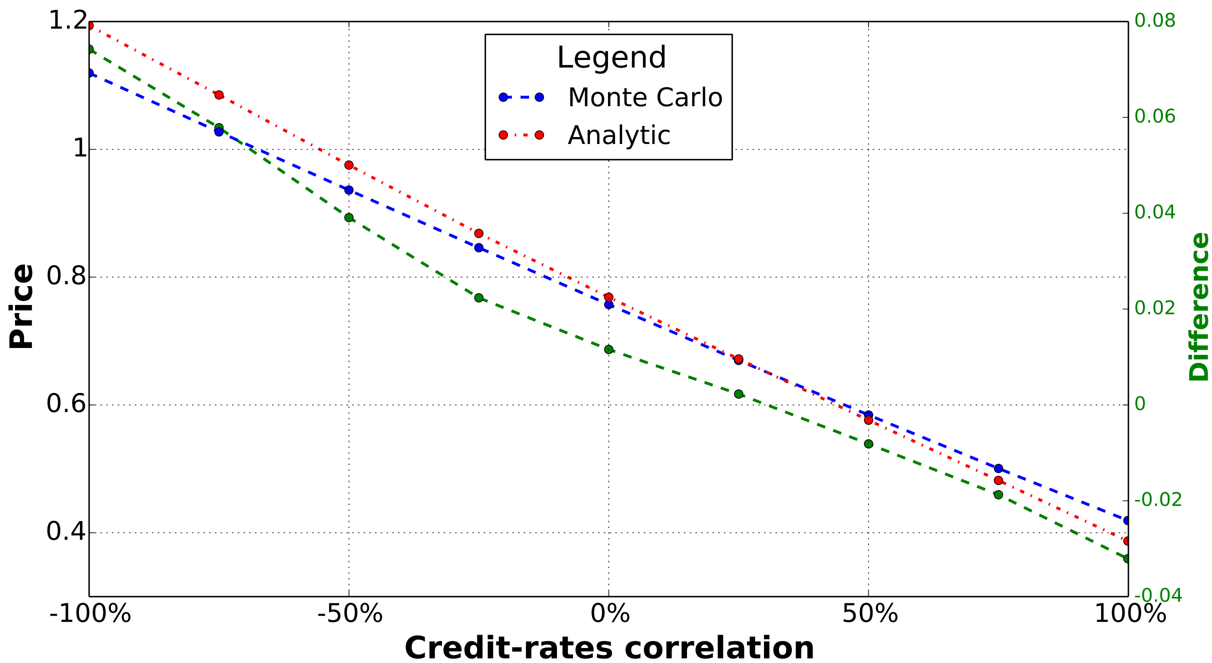

Figure 2: PV dependence of interest rate swap insurance on rates-credit correlation level.

Finally the impact of credit volatility is examined in the presence of a 30% correlation (since there is no impact in the absence of correlation). Results are illustrated in Fig. 3. The agreement is seen to be excellent for credit volatilities up to a rather large value of 80%. The slight discrepancy between the gradients of the two graphs is attributable to the same cause as in the previous case considered.

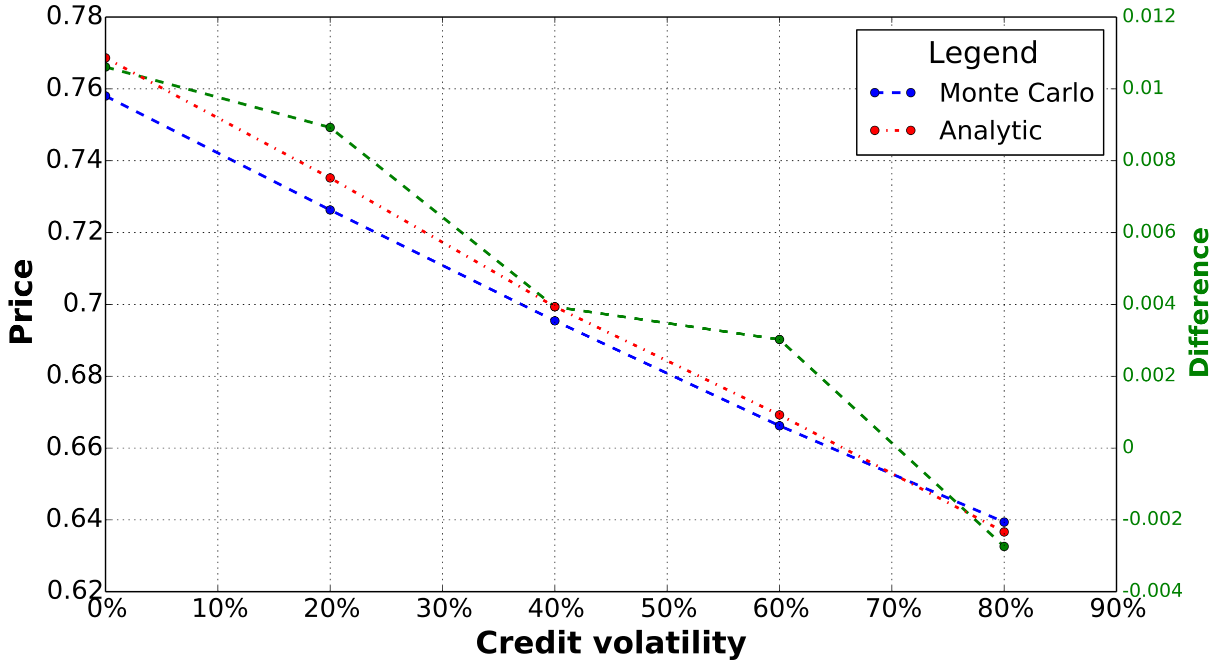

Figure 3: PV dependence of interest rate swap insurance on credit volatility level.

A Asymptotic Pricing

Insurance Pricing

To solve (3.9) for $f(x, y, t)$ subject to (4.5), we must first determine the region in (x, τ) space within which the exposure is positive. Since the exposure is clearly a monotonically *increasing* function of x, which is furthermore negative in the limit as $x \to -\infty$ but positive for large enough x, we infer, for fixed τ, the existence of a critical value of x above which the exposure will be positive. In other words, there exists $x^*(\tau)$ such that

$$x > x^*(\tau) \Leftrightarrow P(x, \tau) > 0.$$

The positive exposure function in (4.5) can on this basis be rewritten as

$$P(x, \tau) = \sum_{i=1}^{n} \left(V_L^{(i)}(x, \tau) - c\Delta_i V_F^{t_i}(x, \tau) \right) \mathbb{1}_{\tau < t_i,\, x > x^*(\tau)}. \tag{A.1}$$

If we denote by $P_0(x, \tau)$ the approximation to the r.h.s. of (4.5) obtained by substituting in the first order approximations (4.2) and (4.4), we can calculate $x^*(\tau)$ to leading order for a given τ as

$$x^*(\tau) \sim \inf\{x \mid P_0(x, \tau) > 0\} \tag{A.2}$$

with errors $= \mathcal{O}(\epsilon_r)$. We will look therefore to calculate $f(\cdot)$ correct to $\mathcal{O}(\epsilon_r \epsilon_\lambda)$, making use of the results in §4.1. The linearity of the positive exposure function and the fact that $f(x, y, t)$ satisfies the homogeneous final condition (4.6) then allow us to obtain the fair price of insurance by solving (3.9) term by term with the help of our asymptotic Green's function. In fact we shall only be interested in the value of $f(\cdot)$ at $t = 0$ (with also $x = y = 0$), so we shall make this simplification from the outset.

Thus we solve (3.9) asymptotically for $t = 0$ with $P(x,\tau)$ taken in turn to be each of $V_L^{(i)}(x,\tau)$ and $V_F^{t_i}(x,\tau)$, contingent on $x > x^*(\tau)$, denoting the results by $f_L^{(i)}$ and $f_F^{t_i}$, respectively. The aggregate cost of insurance is then given by

$$f(0,0,0) \sim \sum_{i=1}^{n} \left(f_L^{(i)} - c\Delta_i f_F^{t_i} \right) \tag{A.3}$$

Fixed Flow

Starting with the fixed leg and making use of the expression (4.2), we see that calculation of $f_F^{t_i}$ to leading order requires use only of the leading order Green's function

$$G_{0,0}^*(x,y,t;\xi,\eta,v) = B(t,v)\frac{\partial^2}{\partial\xi\partial\eta}N_2(\xi - x, \eta - y; R(t,v)), \quad t < v \tag{A.4}$$

where $N_2(x,y,z;R(t,v))$ is a bivariate Gaussian probability distribution function with mean $\mathbf{0}$ and covariance matrix

$$R(t,v) := \begin{pmatrix} I_x(t,v) & I_{xy}(t,v) \\ I_{xy}(t,v) & I_y(t,v) \end{pmatrix}. \tag{A.5}$$

Setting $t = 0$ and carrying out the integration over ξ and η, we obtain

$$\begin{aligned} f_F^{t_i} &\sim \int_0^{t_i} B(0,v)D(v,t_i)\tilde{\lambda}(v) \int_{x^*(v)}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}_y\left(e^{-\alpha_\lambda v} y_v\right) \frac{\partial^2}{\partial\xi\partial\eta} N_2(\xi,\eta;R(0,v))\, d\eta\, d\xi\, dv \\ &\sim \int_0^{t_i} B(0,v)D(v,t_i)\tilde{\lambda}(v) N(-d_1(x^*(v),v,v))\, dv, \end{aligned} \tag{A.6}$$

with relative error $= \mathcal{O}(\epsilon_r + \epsilon_\lambda)$, where

$$d_1(x,v) := \frac{x - e^{-\alpha_\lambda v} I_{xy}(0,v)}{\sqrt{I_x(0,v)}}. \tag{A.7}$$

Libor Flow

Considering next $f_L^{(i)}$, we see that from the form of (4.4) that this has two contributions, one implicitly $\mathcal{O}(\epsilon_r)$ and the other explicitly so. Let us therefore write

$$f_L^{(i)} = f_{L1}^{(i)} + \epsilon_r f_{L2}^{(i)} \tag{A.8}$$

in obvious notation. The first term is, for a given term structure of interest rates, equivalent to a fixed flow at t_i given by the forward Libor rate, whence we deduce

$$f_{L1}^{(i)} \sim \left(D(t_{i-1},t_i)^{-1} - 1\right) f_F^{t_i} \tag{A.9}$$

with relative error $= \mathcal{O}(\epsilon_r + \epsilon_\lambda)$. For the other term, we have

$$\begin{aligned} f_{L2}^{(i)} &\sim D(t_{i-1},t)^{-1} \int_0^{t_i} B(0,v)D(v,t_i)\tilde{\lambda}(v) \int_{x^*(v)}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}_y\left(e^{-\alpha_\lambda v} y_v\right) \frac{\partial^2}{\partial\xi\partial\eta} N_2(\xi,\eta;R(0,v)) \\ &\qquad \int_{t_{i-1}}^{t_i} \left(\mathcal{E}_x\left(F_\beta(u) x_{v\wedge t_{i-1}}\right) - 1\right) \tilde{r}(u)\, du\, d\xi\, d\eta\, dv \\ &\sim D(t_{i-1},t_i)^{-1} \int_0^{t_i} B(0,v)D(v,t_i)\tilde{\lambda}(v) \int_{t_{i-1}}^{t_i} \tilde{r}(u) \\ &\qquad \left(\exp\left(F_\beta(u) e^{-\alpha_\lambda v} I_{xy}(0, v\wedge t_{i-1})\right) N\left(-d_2(x^*(v),u,v,v\wedge t_{i-1})\right) - N\left(-d_1(x^*(v),v)\right)\right) du\, dv, \end{aligned} \tag{A.10}$$

where

$$d_2(x,u,v,w) := d_1(x,v) - \frac{F_\beta(u) I_x(0,w)}{\sqrt{I_x(0,v)}}. \tag{A.11}$$

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